APPLICATIONS TO FLUID MECHANICS: WATER WAVE PROPAGATION

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Summary

Fluid Mechanics is the discipline within the broad field of applied mechanics concerned with the behavior of fluids and gases in motion or at rest. As such it encompasses a vast array of problems that may vary from large scale geophysical flows to the very small scale study of blood flow in capillaries. Within the geophysical flows, the modeling of ocean waves is of great interest for several fields of science and engineering, i.e., oceanography, underwater acoustics, coastal and ocean engineering or navigation.

In this chapter we show that applied mechanics principles such as conservation of mass,

momentum and energy are the foundations of the mathematical modeling of wave propagation. Short wave (T<30 s) propagation is divided into phase-averaged and phase-resolving models. We show the main differences between both approaches by describing the equations and presenting modeling applications at a specific site.

1. Introduction

Coastal dynamics is mostly dominated by surface waves. Waves are disturbances of the equilibrium state in any body of material, in this case a fluid, which propagate through that body over distances and times much larger than the characteristic wave lengths and periods of the disturbances transferring energy and momentum. Everyone interested in ocean processes is aware of the fact that there is an important variety of waves occurring in the oceans and along our shores which can be classified according to their period, wave length or nature of the forcing generating the free surface disturbance.

Tides are the cyclic rising and falling of the ocean surface generated by the forces resulting from the interaction between the earth-moon-sun system. Tides' main constituent periods range from 12 to 24 hours and their wave lengths accordingly vary from hundreds to a few thousands kilometers. With slightly shorter wave periods and wave lengths storm surges are large-scale offshore elevations of the ocean surface associate with a low atmospheric pressure system and high wind speeds. The space and time scales of storm surges are closely correlated with those of the generating storm, i.e. in the order of hundreds of kilometers and one or two days. The combined effect of low pressure, persistent winds and shallow bathymetry may result in catastrophic flooding due to the piling of water against the shore. Even more devastating can become Tsunami waves, a series of waves smaller in wave length and period than storm surges and created when a body of water is rapidly displaced due to earthquakes, land or submarine slide, volcanic eruptions or explosions. The 2004 Indian Ocean Tsunami was generated by an undersea earthquake which triggered a series of devastating tsunamis along the shores of Indonesia, Sri Lanka, India, Thailand and other countries with waves up to 30 m and killing large numbers of people and inundating coastal communities. Tsunami periods may vary from tenth's of minutes to hours and may reach a propagation speed (so-called celerity) up to 800 km/h.

In the range between 30 seconds and a few minutes energy in the ocean is present in the ocean in the form of infragravity waves. Infragravity waves are generated by groups of wind-generated waves and are very relevant in the surf zone where they may dominate beach run-up or be responsible for part of the sediment transport. Furthermore, infragravity waves are the main forcing mechanism of harbor resonance.

Wind generated waves are dominant in the ocean. Usually known as gravity waves, they have wave periods shorter than 30 s and wave lengths in the order of hundreds of meters.

When wind waves are generated by a distance storm, they usually consist of a wide range of wave frequencies and can be described as irregular and short-crested. The wave component with a higher wave frequency propagates at a slower speed than those with lower wave frequencies. As they propagate across the continental shelf towards the coast, long waves lead the wave group and are followed by short waves. When they leave the generation area they become regular and long-crested. In the deep water, wind generated waves are not affected by the bathymetry. Upon entering shoaling waters, however, they are either refracted by bathymetry or current, or diffracted around abrupt bathymetric features such as submarine ridges or canyons. A part of wave energy is reflected back to the deep sea. Continuing their shoreward propagation, waves lose some of their energy through dissipation near the bottom. Nevertheless, each wave profile becomes steeper with increasing wave amplitude and decreasing wavelength. Because the wave speed is proportional to the square root of the water depth in very shallow water, the front face of a wave moves at a slower speed than the wave crest, causing the overturning motion of the wave crest. Such an overturning motion usually creates a jet of water, which falls near the base of the wave and generates a large splash. Turbulence associated with breaking waves is responsible not only for the energy dissipation but also for the sediment movement in the surf zone opening new disciplines such as sediment transport or coastal morphology.

Understanding and predicting the generation of waves and their transformation induced by coastal features or artificial structures are key issues in coastal science and engineering. In the following it will be shown that Fluid Mechanics is crucial for understanding and modeling waves.

In this chapter we will focus on wind waves but the same Fluid Mechanics principles apply to other types of waves previously described.

2. Classification of Wave Models

To date, two basic kinds of wave models can be distinguished: phase-resolving and phase-averaged models.

Even if wind waves are random in nature which may require a statistical approach, under certain circumstances waves can be described using a fully deterministic modeling based on hydrodynamics conservation laws, i.e. conservation of mass, momentum and energy. Using these equations a detailed description of the waves can be given by estimating free surface evolution, velocity and acceleration fields or wave-induced pressure which may be used to calculate forces and moments on structures. The application of phase-resolving numerical models, which require 10 ~ 100 time steps for each wave period, is still limited to relatively small areas, O (1 ~ 10 km) and to relatively short periods of time (10 to hundreds wave periods), and is usually oriented towards the evaluation of wave propagation close to shore, wave agitation in harbors, wave and structure interaction or surf zone hydrodynamics. These models are also used to drive sediment transport and shore morphodynamics models.

For larger length scales, in the order of hundreds of kilometers and hours to year’s time scales the phase-resolving approach is not used since the amount of information required to describe waves would be overwhelming and computationally not affordable. Moreover, a detailed description of the kinematics and dynamics associated to waves is only necessary for certain practical applications at small scales (i.e. it’s not necessary to have a detailed resolution of wave kinematics in the North Atlantic to address a wave agitation problem in a harbor in the North of Spain).

Phase-averaged models, based on the energy balance equation are more relax in the spatial resolution and can be used in much larger regions. The energy balance equation gives the rate of change of the sea state represented by the spectrum that is based on the idea that the profile of ocean waves can be evaluated as the superposition of a high number of harmonic waves, each with its own amplitude, frequency, wave length, direction and phase. The spectrum does represent statistical characteristics of the waves but does not provide detailed information of wave kinematics or dynamics. Moreover, the energy balance equation does include source functions of the generation of waves by wind and provides also a way to introduce wave dissipation due to several different mechanisms.

3. Phase-Averaged Models

The introduction of the concept of a wave spectrum was a key milestone in the development of wave modeling. The wave spectrum at a certain location describes the average sea state (time period of 15-30 min of actual waves for which statistical stationarity of wave characteristics is assumed) in a finite area around that location. The spectrum only gives information on the energy distribution of the waves and not on the phase of the individual waves. Therefore, models based on the energy balance equation, which are able to predict the spectrum at a certain location in the ocean taking into account effects of wave generation (by wind), wave-wave interaction and dissipation (by white-capping) are so-called phase-averaged models.

The spectral energy balance equation is given by

$$\frac{\partial E(f,\theta,x,y,t)}{\partial t}+\frac{\partial C_{g,x}E(f,\theta,x,y,t)}{\partial x}+\frac{\partial EC_{g,y}(f,\theta,x,y,t)}{\partial y}=F(f,\theta;x,y,t) \quad (1)$$

where the left-hand side of the equation represents the rate of change of the energy density of each wave component, $E(f,\theta,x,y,t)$, $C_{g,x}$ and $C_{g,y}$ are the *x*- and *y*-components of the group velocity. $F(f,\theta;x,y,t)$ is the so-called source term which includes the processes of wave generation by wind, wave-wave nonlinear interactions and dissipation by white-capping, a wave breaking process that occurs in deep water. For large-scale applications, this equation is usually transferred to a spherical coordinate system.

Figure 1. Location of the Port of Almeria

This so-called wave generation models are currently being used in operational systems and many times provide input information to run phase-resolving models in more limited areas. The most extended wave generation models are WAVEWATCH III (Tolman 1991, 1999) the WAM model (WAMDIG 1988, Komen et al. 1994).

Figure 1, shows the location of our applications in this chapter, the Port of Almeria. Almeria is located at the southeast part of Spain at the Mediterranean coast. The dominant and prevailing winds come from the southwest, and the area is characterized by a small tidal range of 0.6 m and a relatively strong wave climate. The water depths inside the harbor are in the range of 10 m. The harbor includes several breakwaters and quays.

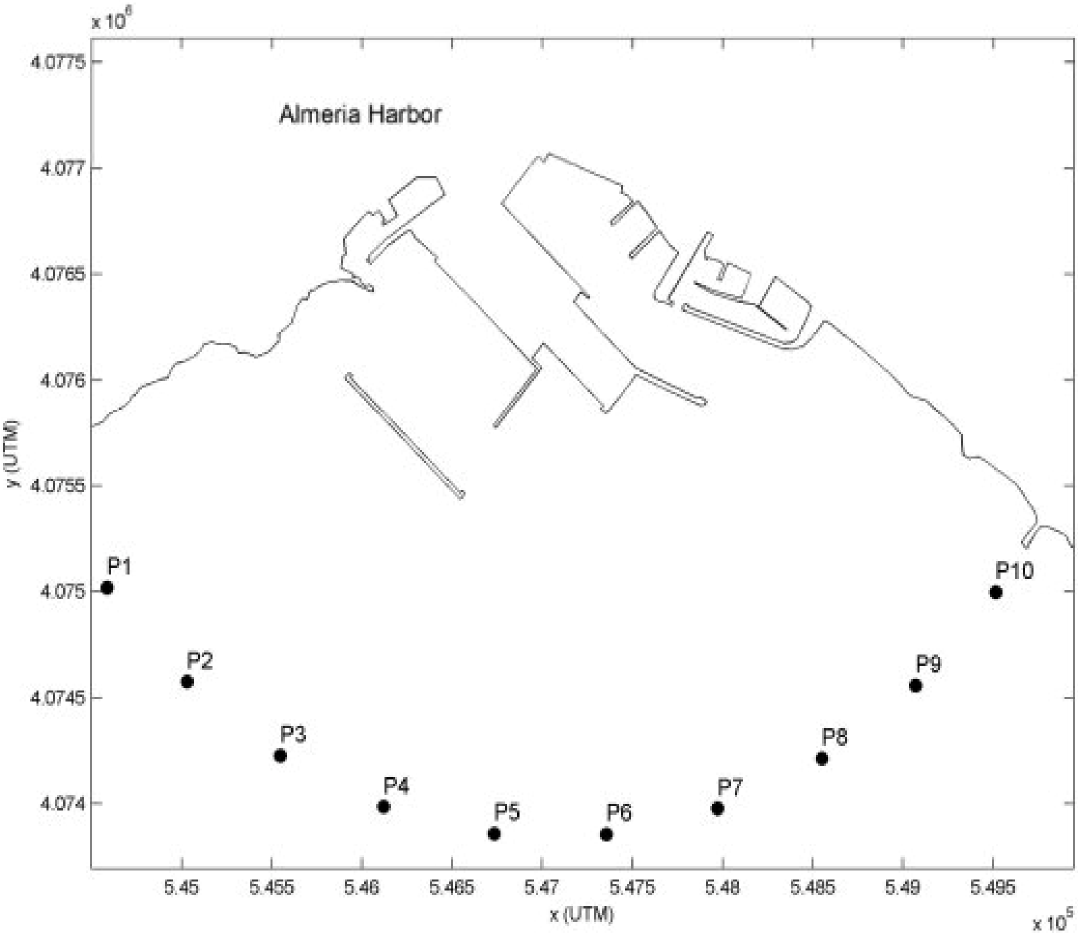

Figure 2. Port of Almeria geometry

Figure 2, shows the geometry of the harbor. As can be seen in this geometry the most offshore quay is protected by a detached breakwater to be constructed. Points P1 to P10, represent the locations where the input spectra at the offshore boundary of the phase-resolving numerical model, later explained in this chapter, are located. As will be shown, these spectra result from the wave propagation carried out using a phase-averaged model based on (1).

In order to evaluate how waves reach the Port of Almeria the wave climate transformation has to be evaluated. Depending on the track of the storms the wave may reach the harbor from different locations. Since the geographical domain is large a phase-average model (WAM) is used to obtain the wave climate in deep water. The model solving Eq. (1) allows the nesting of different meshes to increase spatial resolution while approaching the harbor. Using spectra provided by a west Mediterranean mesh (not shown) as an input for Mesh A, two additional meshes B and C, provide a high resolution information on wave spectra close to the harbor (50 m x 50 m grid). These meshes are used to run the SWAN (Simulating Waves Nearshore) model, a shallow water version of the wave generation model, WAM.

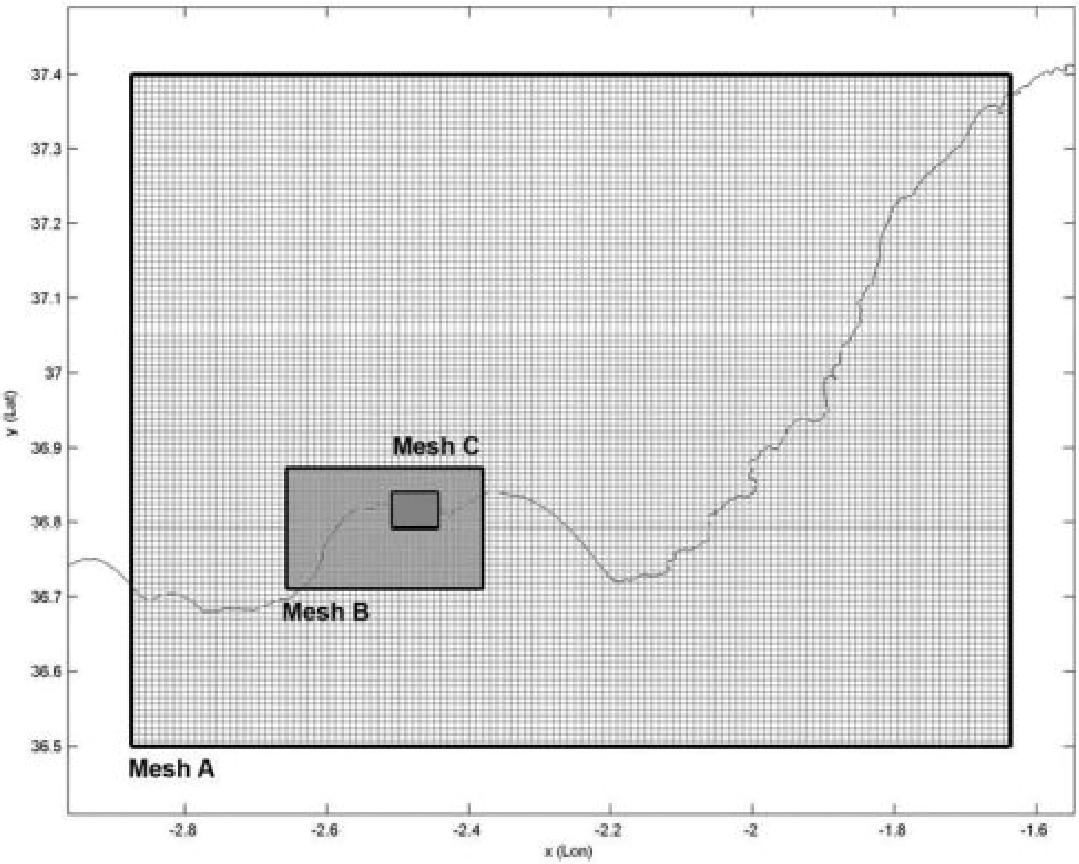

Figure 3. Location and size of the nested meshes used for the phase-averaged wave propagation model. Mesh A (500 m x 500m), Mesh B (100 m x100 m), Mesh C (50 m x 50 m)

Figure 4 a, b and c shows the results for given offshore wave conditions. The three plots show arrows indicating the direction of the incoming waves and a grey-scale indicating wave height.

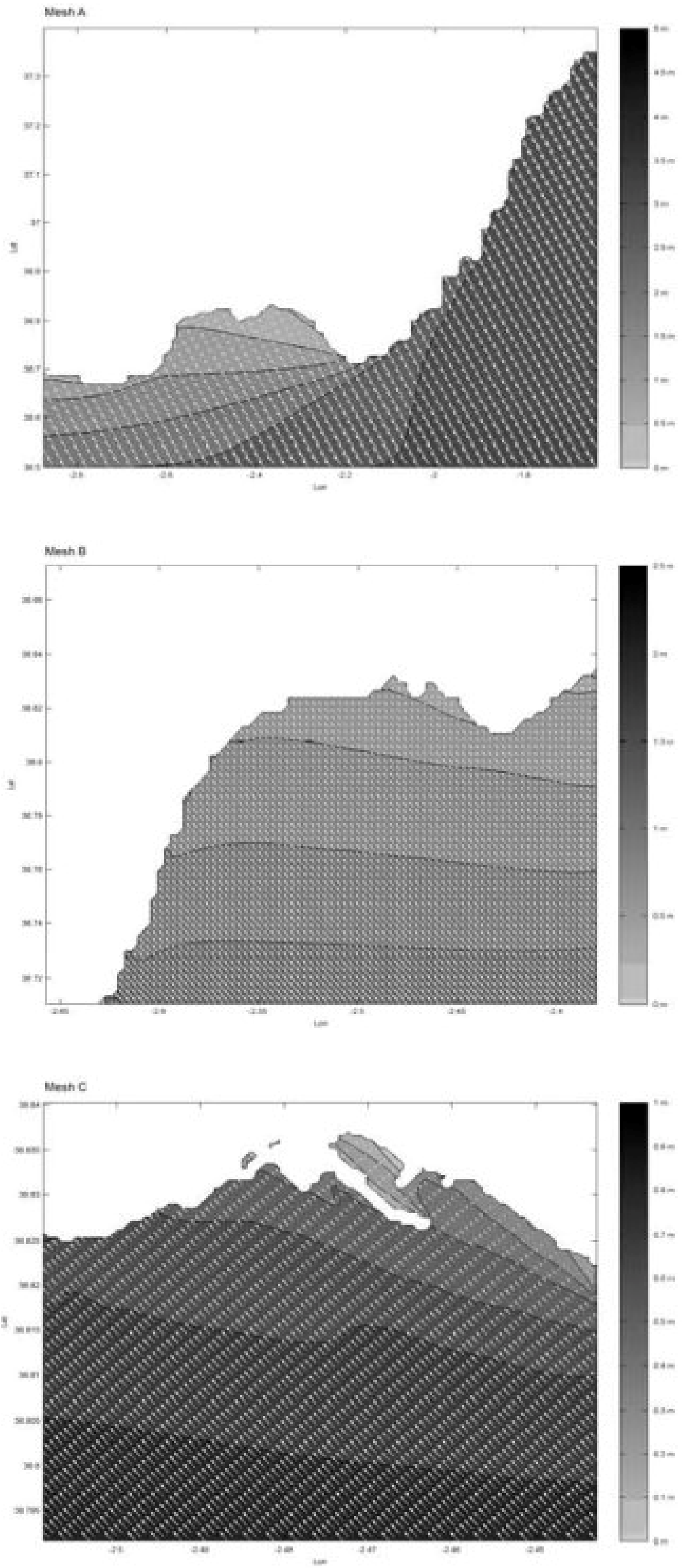

Figure 4. Wave height and direction for wave propagation calculated using the phase-averaged model SWAN at the Port of Almeria. Results are shown for the nested meshes A, B and C and waves arriving from the southeast.

For the case presented waves arrive from the southeast perpendicular to the harbor main entrance, introducing wave agitation inside the harbor. Wave height is reduced close to the coastline. Computed significant wave height at the harbor entrance is about 0.3 m. Wave diffraction induced by the breakwaters will introduces additional wave height reduction resulting in a limited wave height at the quays and docks.

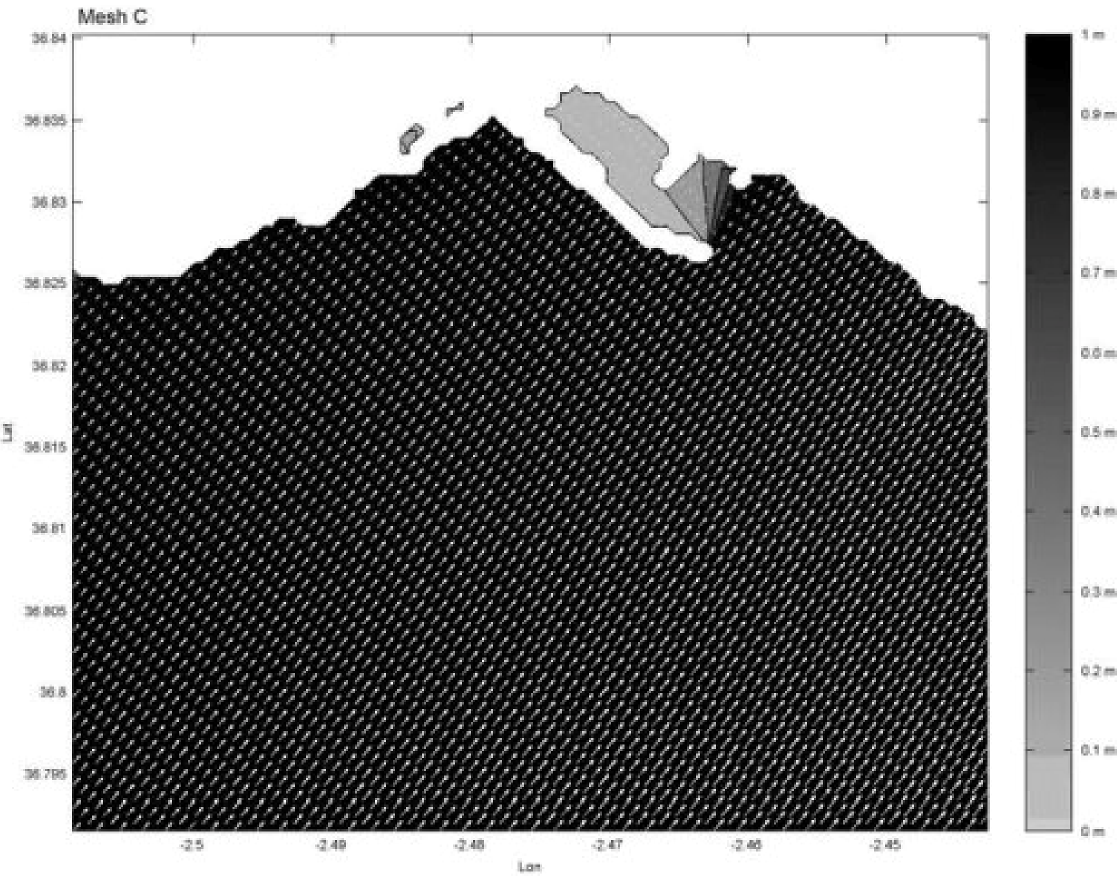

Figure 5. Wave height and direction for wave propagation calculated using the phase-averaged model SWAN at the Port of Almeria. Results are shown for mesh C and waves arriving from the southwest.

For waves propagating from the west, the waves arrive to the port impinging normally on the outer breakwater with a very limited wave height variation along the coast. The characteristic significant wave height is about 1 m. Due to wave diffraction the wave inside the harbor is considerably reduced. However, this modeling does not provide information to solve the waves' phase and therefore, no detailed information on several physical magnitudes of great importance to analyze harbor operations and structures' stability.

4. Phase-Resolving Models

4.1. Introduction

In principle, water wave motions can be modeled by the Navier-Stokes equations for incompressible Newtonian fluids, which represent the conservation of mass and momentum. Free surface boundary conditions, ensuring the continuity of stress tensor across the free surface and the free surface is a material surface, are necessary in determining the free surface location. Both the Navier-Stokes equations and the free surface boundary conditions are nonlinear. Consequently, even when the viscous and turbulence effects can be ignored, the computational effort required for solving a truly three-dimensional wave propagation problem, which has a horizontal length scale of over hundreds or more wavelengths, is too large to be employed in real applications at this time. Consequently, wave theories have been evolving in parallel with computer power. The first wave models tried to simplify the general formulation of the problem by introducing certain assumptions that made the problem solvable.

4.2. Governing Equations for Water Waves

Consider the Cartesian coordinate system as shown in Figure 6. The free surface is given by $z=\eta(x,y,t)$ and the rigid, impermeable bottom is given by $z=-h(x,y)$. Assuming the water to be incompressible and a non-viscous flow, the governing equations are the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{at} \quad -h(x,y)\le z\le \eta(x,y,t) \tag{2}$$

the equations of motion are

$$\begin{aligned} \frac{Du}{Dt}&=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{at} \quad -h(x,y)\le z\le \eta(x,y,t)\\ \frac{Dv}{Dt}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}\\ \frac{Dw}{Dt}&=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \end{aligned} \tag{3}$$

which are known as the Euler equations.

To solve the wave problem we need to solve the continuity and momentum equations above for specific boundary conditions. These boundary conditions relate the fluid flow and the boundaries (kinematic boundary conditions) in this case applying at the free surface, bottom and lateral boundaries; but also they have to relate the boundaries with the forces acting on the fluid (dynamic boundary conditions). Being the bottom an impermeable rigid boundary, the dynamic boundary condition only applies at the free surface.

The kinematic boundary condition states that a particle at the boundary does not leave the boundary which can be expressed as

$$w=\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+\mathrm{v}\frac{\partial \eta}{\partial y} \qquad \text{at} \;\; z=\eta(x,y,t) \tag{4}$$

at the free surface and

$$w = -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} \quad \text{at } z = -h \tag{5}$$

at the impermeable bottom

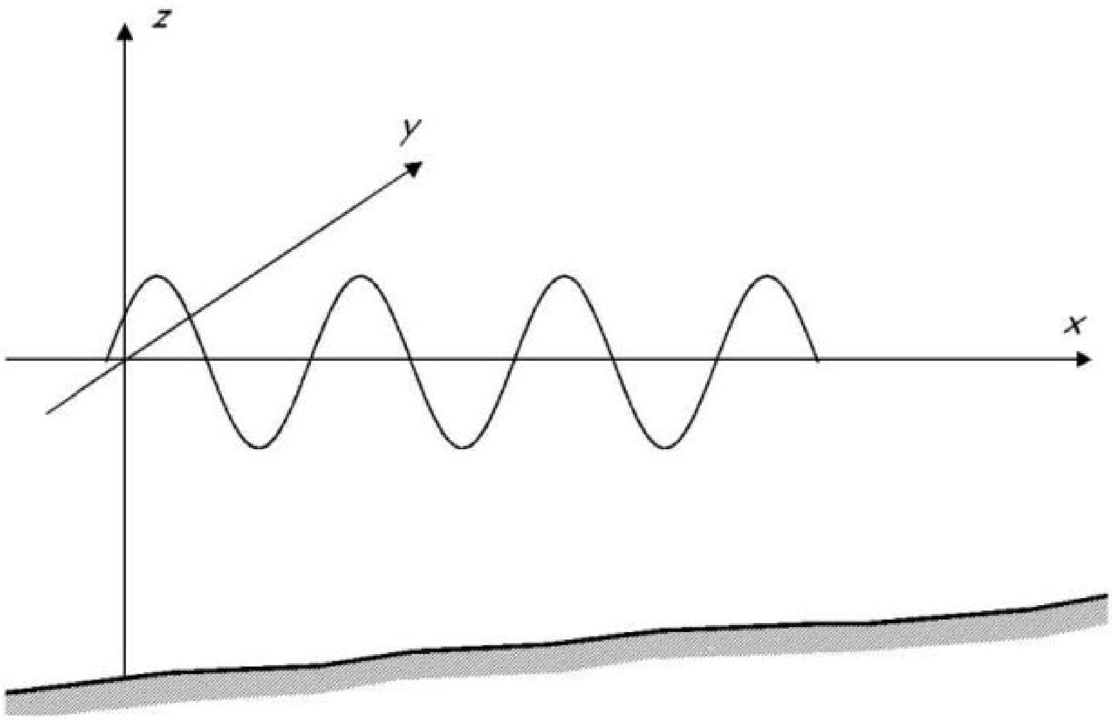

Figure 6. Reference System for Water Wave Propagation

For an interface between two fluids the dynamic boundary condition consists of continuity in the tangential velocities at either side of the interface and continuity of the normal stresses, apart from the contribution of surface tension. Considering that there is no viscosity the dynamic boundary condition is given by

$$p = p_\mathrm{a} + T\left[\eta\right] \quad \text{at } z = \eta(x, y, t) \tag{6}$$

where p_a is the atmospheric pressure and $T\left[\eta\right]$ is the surface tension effect that can be expressed in terms of the first and second derivatives of the free surface.

In order to solve this set of equations initial conditions have to be added. For a non viscous flow, if the rotation is zero (zero vorticity) initially it remains zero and therefore the flow can be considered to be irrotational. Due to irrotationality a scalar velocity potential $\phi(x, y, z, t)$ may be introduced, which is defined such that

$$u = \frac{\partial \phi}{\partial x}; v = \frac{\partial \phi}{\partial y}; w = \frac{\partial \phi}{\partial z} \tag{7}$$

The continuity equation and the relation between velocity field and velocity potential can be combined to yield Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{8}$$

Combining the Euler equations an unsteady Bernoulli equation can be obtained under the irrotational ideal flow hypothesis such that

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(u^2+v^2+w^2\right)+\frac{p}{\rho}+gz=f(t) \tag{9}$$

This equation is valid in the region $-h(x,y)\le z\le\eta(x,y,t)$ and the integration constant $f(t)$ can be eliminated by redefining the velocity potential by

$$\frac{\partial\Phi}{\partial t}=\frac{\partial\phi}{\partial t}-f(t) \tag{10}$$

By combining the Bernoulli equation (9), (10) and (6) a new dynamic free surface boundary condition can be obtained, including the effects of surface tension and atmospheric pressure

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)+\left(\frac{\partial\Phi}{\partial y}\right)+\left(\frac{\partial\Phi}{\partial z}\right)\right]+\frac{p_{\mathrm{a}}}{\rho}-\frac{1}{\rho}T\left[\eta\right]+g\eta=0 \quad \text{at } z=\eta(x,y,t)$$

By writing the kinematic boundary conditions (4) and (5) in terms of the velocity potential the governing equations for irrotational wave motion are given by the Laplace equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \quad \text{at } -h(x,y)\le z\le\eta(x,y,t) \tag{11}$$

and the three following boundary conditions

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]+\frac{p_{\mathrm{a}}}{\rho}-\frac{1}{\rho}T\left[\eta\right]+g\eta=0 \quad \text{at } z=\eta(x,y,t) \tag{12a}$$

$$\begin{aligned}&\frac{\partial\eta}{\partial t}+\nabla\Phi\bullet\nabla\eta=\frac{\partial\Phi}{\partial z} && \text{at } z=\eta(x,y,t)\\ &\frac{\partial\Phi}{\partial t}+\nabla\Phi\bullet\nabla h=0 && \text{at } z=-h(x,y)\end{aligned} \tag{12b}$$

where $\nabla = (\partial / \partial x, \partial / \partial y)$ is a two dimensional vector.

It is customary to express both free surface boundary conditions in terms of a combined free surface condition in Φ so that the free surface displacement η remains only in the definition of the unknown domain where the problem has to be solved. Furthermore, considering the atmospheric pressure to be zero, which means that the pressure in the fluid to be determined is the pressure in excess of the atmospheric pressure and neglecting the surface tension effects the equations can be written as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{at } -h(x,y) \le z \le \eta(x,y,t) \tag{13a}$$

$$\frac{\partial \Phi}{\partial z} + \nabla\Phi \bullet \nabla h = 0 \qquad \text{at } z = -h(x,y) \tag{13b}$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} + \left[\frac{\partial}{\partial t} + \frac{1}{2}\nabla\Phi \bullet \nabla + \frac{1}{2}\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial z}\right]\left\{\left(\frac{\partial \Phi}{\partial z}\right)^2 + \left|\nabla\Phi\right|^2\right\} = 0 \quad \text{at } z = \eta(x,y,t) \tag{14}$$

Using the appropriate lateral boundary conditions, once the solution of Φ has been determined; the free surface elevation can be evaluated using the dynamic free surface boundary condition

$$\eta = -\frac{1}{g}\left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right\}\right] \quad \text{at } z = \eta(x,y,t) \tag{15}$$

Once $\eta(x,y,t)$ and $\Phi(x,y,z,t)$ has been solved, the velocities follow from the definition of the velocity potential and the expression for pressure can be obtained from the unsteady Bernoulli equation.

The governing equations presented here cannot be solved exactly, the principal difficulty being the fact that one of the unknowns in the problem, i.e., the free surface, η, is needed to define the domain where the equations have to be solved. The only way to solve this problem is by deriving approximate models. The selection of the right approximate model depends on the space and time scales of the problem considered and has to be determined depending on the values of non-dimensional parameters such as the relative depth given by $\left(h/L\right)$ or (kh) and the relative wave height (H/h) or wave steepness (H/L), where h stands for water depth, H is the wave height, L is the wavelength and $k = 2\pi/L$ the wave number.

4.3. Linear Wave Theory

The simplest approximate model is the linear wave theory also known as Airy or

Stokes-order one theory. In this theory linearization is performed by ignoring terms of quadratic and higher order by assuming that wave height (H/h) and wave steepness (H/L) are small or in terms of the Ursell parameter, if $U_{\mathrm{r}} = \left((A/h)/(kh)^2\right) \ll 1$, $A(\approx H/2)$ being the wave amplitude. This assumption is extremely convenient because nonlinear terms are neglected and the boundary conditions referred to a known geometric surface, so that the domain is perfectly defined a priori. This approach has been proven to be very successful in many situations and the assumption of linearity permits the superposition of solutions and therefore the definition of complex motions also present in nature.

To simplify the solution of the problem in the following linear wave theory is considered for two-dimensional waves propagating on a horizontal bottom.

Considering Eqs. (11), (12a) and (12b), neglecting surface tension, the second horizontal dimension and assuming the atmospheric pressure to be zero the following equations are obtained

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad \text{at } -h \le z \le \eta(x,t)$$

and the three following boundary conditions

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] + g\eta = 0 \quad \text{at } z = \eta(x,t)$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial z} \qquad \text{at } z = \eta(x,t)$$

$$\frac{\partial\Phi}{\partial z} = 0 \qquad \text{at } z = -h$$

Please note that Eq. (13b), known as the kinematic bottom boundary condition, has been simplified by assuming constant water depth.

For this set of equations we still have the problem of the presence of nonlinear terms and the domain defined in terms of one of the unknowns. By taking Taylor expansions around a fixed level, usually $z = 0$, which is known, and dropping all the terms of quadratic or higher order the problem is reduced to

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad \text{at } -h \le z \le 0 \tag{16}$$

and the three following boundary conditions

$$\frac{\partial \Phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0 \tag{17}$$

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\partial \Phi}{\partial z} && \text{at } z = 0 \\ \frac{\partial \Phi}{\partial z} &= 0 && \text{at } z = -h \end{aligned} \tag{18a,b}$$

Adding two additional conditions, i.e. the solution to be periodic in space and time

$$\begin{aligned} \Phi(x+L, z, t) &= \Phi(x, z, t) \\ \Phi(x, z, t+T) &= \Phi(x, z, t) \end{aligned} \tag{19a,b}$$

where L is the wave length and T the wave period.

The Laplace's equation can be solved by using the method of separation of variables and introducing a separation variable k which can be identified with the wave number such that $k = 2\pi / L$.

By applying the boundary conditions the solution for the velocity potential corresponding to a progressive wave is given by

$$\Phi(x, z, t) = \frac{H}{2} \frac{g}{\omega} \frac{\cosh k(h+z)}{\cosh kh} \sin(kx - \omega t) \tag{20}$$

where $\omega = (2\pi / T)$ is the wave frequency. Note that the vertical motion is separated from the motion in horizontal space which in this case is given by a periodic function. The argument of the periodic function $(kx - \omega t)$ is the so-called phase of the wave.

From the dynamic boundary condition (17) it follows that the free surface elevation is expressed as

$$\eta(x, t) = \frac{H}{2} \cos(kx - \omega t) \tag{21}$$

Values of the phase which correspond to cosine values 1 or -1, represent wave crests and troughs, respectively.

In order to close the solution a dispersion relation is derived from the kinematic free surface boundary condition giving the following equation which relates wave frequency, ω, or wave period, T water depth, h and wave number or wave length, L, as

$$\omega^2 = gk \tanh kh \tag{22}$$

This equation can be re-written in terms of the wave propagation speed or wave celerity,

C as

$$C^2 = \frac{\omega^2}{k^2} = \frac{L^2}{T^2} = \frac{g}{k}\tanh kh \tag{23}$$

This solution can be easily extended to two horizontal dimensions for the case of plane waves by expressing the scalar wave number as a wavenumber vector and defining the wave phase as $(\vec{k}\cdot\vec{x}-\omega t)$ such that

$$\Phi(x,y,z,t) = \frac{H}{2}\frac{g}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\sin(\vec{k}\cdot\vec{x}-\omega t) \tag{24}$$

$$\eta(x,y,t) = \frac{H}{2}\cos(\vec{k}\cdot\vec{x}-\omega t) \tag{25}$$

with $\vec{k} \equiv (k_x, k_y) = (k\cos\theta, k\sin\theta)$ the wavenumber vector and θ the wave angle of incidence. In the dispersion equation, now $k \equiv \left|\vec{k}\right|$.

By using the relations (7) and (24) the velocity field can be obtained

$$u = \frac{\partial\Phi}{\partial x} = \frac{H}{2}\frac{gk_x}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(\vec{k}\cdot\vec{x}-\omega t)$$
$$v = \frac{\partial\Phi}{\partial y} = \frac{H}{2}\frac{gk_y}{\omega}\frac{\cosh k(h+z)}{\cosh kh}\cos(\vec{k}\cdot\vec{x}-\omega t)$$
$$w = \frac{\partial\Phi}{\partial z} = \frac{H}{2}\frac{gk}{\omega}\frac{\sinh k(h+z)}{\cosh kh}\sin(\vec{k}\cdot\vec{x}-\omega t)$$

and the pressure follows from Bernoulli equations by dropping the nonlinear terms

$$p(x,y,z,t) = -\rho gz - \rho\frac{\partial\Phi}{\partial t} = -\rho gz - \rho\frac{\cosh k(h+z)}{\cosh kh}\eta(x,y,t)$$

The pressure has two different contributions, the hydrostatic component and the dynamic component which is proportional to the free surface displacement.

For further applications, it is necessary to explain that the energy associated to the wave travels at the group celerity, C_g, which is related to the wave celerity by the following relation

$$C_g = \frac{1}{2}\left[1+\frac{kh}{2\sinh 2kh}\right]C$$

Note that in the two horizontal dimensions space, both the celerity and group celerity are vectors in the direction of wave propagation. The linear solution presented provides an excellent means to find new solutions based on the principles of superposition representing more realistic wave free surface such as standing or quasi-standing waves (resulting from the interaction of incident and reflected waves); short-crested waves (resulting from the interaction of wave propagating in different directions) or wave groups (resulting from wave propagating in the same direction with slightly different frequencies). Moreover, the superposition of a large number of wave components with different wave amplitudes, frequencies and directions may be a good approximation to represent a real wave spectrum as the ones calculated using phase-averaged models.

However, at this point, the solution presented has been found assuming a horizontal bottom, an important shortcoming for real applications.

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Bibliography

Battjes, J.A. and Janssen, J.P.F.M. (1978). Energy loss and set-up due to breaking of random waves. *Proceedings of the 16th International Coastal Engineering Conference, ASCE*, 569-587. [One of the first derivations of a semi-empirical model for breaking waves]

Berkhoff, J. C. W. (1972). Computation of combined refraction and diffraction. *Proceedings of the 13th International Coastal Engineering Conference, ASCE*, 471-490. [It laid the foundation of the mild-slope equation, one of the most popular equations for linear wave propagation in large domains]

Berkhoff, J. C. W. (1976). Mathematical models for simple harmonic linear water waves; wave refraction and diffraction. *PhD thesis, Delft Technical University of Technology.* [Detailed information on the derivation and validation of the mild slope equation can be found in this thesis]

Chen, Q., Madsen, P. A., Schaffer, H. A. and Basco, D. R. (1998). Wave-current interaction based on an enhanced Boussinesq approach. *Coastal Engng.*, 33, 11-39. [One of the first derivations of Boussinesq-type equations for the combined analysis of waves and currents in shallow waters]

Chen, Y. and Liu, P. L.-F. (1995). Modified Boussinesq equations and associated parabolic models for water wave propagation. *J. Fluid Mech.*, 288, 351-381. [It contains an alternative derivation to Nwogu (1993) extended Boussinesq equations and a parabolic version of these new equations]

Dalrymple, R.A. and Rogers, B.D. (2006). Numerical modeling of water waves with the SPH method. *Coastal Engineering*, 53, 141-147. [Smoothed Particle Hydrodynamics (SPH) is introduced with several applications in water wave propagation].

Dalrymple, R.A., Kirby, R.T. and Hwang, P.A. (1984). Wave diffraction due to area of energy dissipation. *J. of Waterway, Port, Coastal, and Ocean Engineering*, ASCE., 110, No. 1., 67-79. [Presents concepts and formulation on the role of wave energy dissipation that have been incorporated into several propagation models]

Elgar, S. and Guza, R. T. (1985). Shoaling gravity waves: comparisons between field observations, linear theory and a nonlinear model. *J. Fluid Mech*., 158, 47-70. [Includes the first comparisons of a nonlinear model based on Boussinesq-type equations with observed statistics of non-breaking waves in the field]

Garcia, N., Lara, J.L. and Losada, I.J. (2004). 2-D Numerical analysis of near-field flow at low-crested permeable breakwaters. *Coastal Engineering*, ELSEVIER, 51, no. 10, 991-1020. [First applications of RANS equations considering wave breaking on permeable submerged structures including a detailed validation with experimental data]

Goring, D. G. (1978). Tsunamis - the propagation of long waves onto a shelf. *Ph.D. dissertation, California Institute of Technology, Pasadena, CA*. [It contains the modeling of solitary waves shoaling on a slope and compares numerical and experimental data]

Hsu, T.-J., Sakakiyama, T., Liu, P.L.-F. (2002). A numerical model for wave motions and turbulence flows in front of a composite breakwater. *Coastal Engineering*, 46, 25-50. [Includes the derivation of the Volume-Averaged/Reynolds Averaged Navier-Stokes (VARANS) for porous flow]

Jaw, S.Y. & Chen, C.J. (1998a). Present status of second-order closure turbulence model. I: overview. J. Engineering Mechanics, 124, 485-501. [An overview of the second-order closure turbulence models is presented in this paper].

Jaw, S.Y. & Chen, C.J. (1998b). Present status of second-order closure turbulence models. II: application. *J. Engineering Mechanics*, 124, 502-512. [Examples of how different turbulence models apply to several examples of complex flows].

Karambas, Th. V. and Koutitas, C. (1992). A breaking wave propagation model based on the Boussinesq equations. *Coastal Engineering*, ELSEVIER, 18, 1-19. [Wave breaking is introduced in the Boussinesq type equations by including a dispersion term to simulate Reynolds stresses. The eddy viscosity is calculated using the mixing length hypothesis]

Kirby J.T. and Dalrymple, R.A. (1983). A parabolic equation for the combined refraction-diffraction of Stokes waves by mildly varying topography. *J. Fluid Mech*., 136, 543-566. [It contains the foundations of one of the most popular phase-resolving models REF/DIF, based on the parabolic version of the mild-slope equation]

Komen, G. J., Cavaleri, L., Donelan, M., Hasselmann, K., Hasselmann, S., Janssen, P. A. E. M. (1994) Dynamics and Modeling of Ocean Waves. *Cambridge University Press*, 532 pp. [Reference book on wave modeling focused on phase-averaged models]

Lara, J.L., Garcia, N. and Losada, I.J. (2006a). RANS modeling applied to random wave interaction with submerged permeable structures. *Coastal Engineering*., 53 (5-6), 395-417, ELSEVIER. [It includes random wave transformation over submerged structures, considering all the relevant 2-D processes, using RANS equations]

Lara, J.L., Losada, I.J., Liu, P.L.-F. (2006b). Breaking waves over a mild gravel slope: experimental and numerical analysis. *Journal of Geophysical Research*, AGU, 111, C11019; doi: 10-1029/2005 JC003374. [First numerical study on wave breaking on permeable bottoms using RANS equations]

Lin, P. and Liu, P. L.-F. (1998). A Numerical Study of Breaking Waves in the Surf Zone, *J. Fluid Mech*., 359, 239- 264. [A seminal paper introducing in coastal engineering the analysis of wave breaking using RANS equations and using a Volume of Fluid technique]

Liu, P. L.-F. (1990). Wave transformation, in the Sea, v. 9, 27-63. [One of the most cited review papers on wave transformation modeling]

Liu, P. L.-F. (1994). Model equations for wave propagation from deep to shallow water. *Advances in Coastal and Ocean Engineering*, v.1, 125-158. [It includes an exhaustive review of how Boussinesq equations have been extended to be valid from deep to shallow water]

Liu, P. L.-F., Lin, P., Chang, K.-A., and Sakakiyama, T. (1999). Wave interaction with porous structures. *J. Waterway, Port, Coastal and Ocean Engrg*., ASCE, 125, (6), 322-330. [First application of a RANS model to analyse wave interaction with permeable structures using a VOF technique]

Liu, P.L.-F., Wu, T.-R., Raichlen, F., Synolakis, C. and Borrero, J.C. (2005). Runup and rundown generated by three-dimensional sliding masses. *J. Fluid Mech*., 536, 107-144 [A 3-D model based on a large-eddy-simulation (LES) approach is developend and validated against large-scale experiments].

Losada, I.J., Gonzalez-Ondina, J.M., Diaz, G., Gonzalez, E.M. (2008a). Numerical simulation of transient nonlinear response of semi-enclosed water bodies: model description and experimental validation. *Coastal Engineering*, ELSEVIER, 55 (1), 21-34. [Extended Boussinesq equations area the basis for a finite element model used to study nonlinear resonance in basins]

Losada, I.J., Lara, J.L., Guanche, R., Gonzalez-Ondina, J.M. (2008b). Numerical analysis of wave overtopping of rubble mound breakwaters. *Coastal Engineering*, ELSEVIER, 55 (1), 47-62. [Includes the description of the RANS model COBRAS-UC and its validation for coastal structures applications]

Madsen, P. A., Murray, R. and Sorensen, O. R. (1991). A new form of the Boussinesq equations with improved linear dispersion characteristics. *Coastal Engineering*, 15, 371-388. [It laid the foundations of the extension of the standard Boussinesq equations to shallow and deep water by improving the dispersion characteristics in deep water]

Nwogu, O. (1993). An alternative form of the Boussinesq equations for nearshore wave propagation. *J. Waterway. Port, Coast. Ocean Engng*, ASCE, 119, 618-638. [For the first time a new form of the standard Boussinesq Eqs. (weakly dispersive-weakly nonlinear) was derived significantly improving their linear dispersion properties and making them applicable to a wider range of water depths]

Peregrine, D. H. (1967). Long waves on a beach. *J. Fluid Mech*., 27, 815-882. [This is the fundamental paper in which the Boussinesq equations were introduced for the first time in water wave propagation modeling]

Pope, S.B. (2000) Turbulent Flows, Cambridge University Press. [Reference book in turbulence]

Schäffer, H.A., Madsen, P.A. and Deigaard, R.A. (1993). A Boussinesq model for waves breaking in shallow water. *Coastal Engineering*, 20, 185-202. [The concept of surface rollers is introduced as a simple description of wave breaking and incorporated in the Boussinesq equations]

Thornton, E.B. and Guza, R.T. (1983). Transformation of wave height distribution. *J. of Geophysical Research*, 88. C10, 5925-5938. [One of the most extended semi-empirical models of wave height distribution under breaking conditions]

Tolman, H. L. (1991). A third-generation model for wind waves on slowly varying, unsteady and inhomogeneous depths and currents. *J. Phys. Oceanogr.* , 21, 782-797. [It laid the foundation of WAVEWATCH-III, a phase-averaged model currently part of many operational systems for wave prediction and the basis for many wave historical data bases]

Tolman, H. L. (2009). User manual and system documentation of WAVEWATCH-III^{TM} version 3.14. NOAA / NWS /NCEP /MMAB Technical Note 276, 194 pp. [Includes governing equations, numerical approaches and recommendations to run the model (public domain software)]

Torres-Feyermouth, A., Losada, I.J., Lara, J.L. (2007). Modeling of surf zone processes on a natural beach using RANS equations. *Journal of Geophysical Research*, AGU, 112, C09014, doi:10.1029/2006JC004050. [First application of a RANS model to simulate hydrodynamics in real beaches. It includes comparisons with field data]

Tsay, T.-K. and Liu, P.L.-F. (1982). Numerical solution of water-wave refraction and diffraction problems in the parabolic approximation. *Journal of Geophysical Research*, 87 (C10), pp. 7932-7940. [The introduction of the parabolic approximation for the mild-slope equation did open the opportunity to propagate waves over large domains for phase resolving models]

WAMDIG (1988). The WAM model - A third generation ocean wave prediction model. *Journal of Physical Oceanography*, 18, 1775-1810. [The foundations of the WAM model one of the most extended phase-averaged models is laid in this paper]

Wei, G., Kirby, J. T., Grilli, S. T. and Subramanya, R. (1995). A fully nonlinear Boussinesq model for surface waves. Part I. Highly nonlinear unsteady waves. *J. Fluid Mech*., 294, 71-92. [One of the first derivations of the fully nonlinear Boussinesq equations opening the possibility to extend the application of the standard Boussinesq equations over a much wider range]

Zelt, J. L. (1991). The run-up of nonbreaking and breaking solitary waves. *Coastal Engng*, 15, 205. [Breaking is parameterized with an artificial viscosity (diffusion) term in the momentum equation, and

bottom friction is modeled with a term quadratic in the horizontal fluid velocity]

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